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Discounted perpetual game call options

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ABSTRACT

The purpose of this paper is to examine the problem of pricing discounted perpetual game call options. In addition to the properties of the American options, the game options give the seller the right to cancel the contract at some chosen from him moment. As a compensation for this, he has to pay some amount above the usual payment. We assume that this penalty payment is a constant. We examine the case without maturity – the exercise can be made in every future moment. We first derive the optimal exercise regions for the buyer and the seller and then calculate the fair option price. Our approach is based on some American style derivatives with a stochastic maturity date.

1. Introduction

The game, or Israeli, options are introduced by Kifer [12]. They are a particular case of the so called Dynkin games, see Dynkin [7]. They appear as a natural extension of the American options. The strong scientific interest in the area of the American style derivatives is documented by the large amount of recent publications – we reffer to Yoon [25], Klimsiak et al. [14], Park and Jeon [21], Le and Dang [18], Balajewicz and Toivanen [1], Gong and Zhuang [11], Zhao and Yang [29], Chen et al. [6], Madi et al. [19], Zaevski [28], Chan [4], Zaevski [27], Pereira and Rodrigues [22], Chen et al. [5], and Zaevski [26]. The main feature of the American options is that the holder has the right to exercise at every moment before the maturity. The game options add the seller's right to cancel the option. The price of this right is the seller's obligation to pay to the buyer some amount above the ordinary option payment, which is called a penalty. We assume that it is a constant η. In that way both the buyer and the seller have the right to stop the contract at every moment before the maturity.

Along with Kifer [12] and Kifer [13], there are several seminal works in the area. Kunita and Seko [16] examine the exercise regions for the game call options. In [17] and Kühn and Kyprianou [15] are used the variational inequalities. In [9] and Ekström [8] is used the method of the so called excessive functions. Emmerling [10] and Yam et al. [24] study the case when the underlying asset admits dividend payments. In [20] game options are discussed from the point of view of the free boundary problems.[1]

In this article we examine the perpetual case and we introduce a discount factor with positive rate $\lambda > 0$. It has a great significance in the practice of financial derivatives markets. The discount factor introduces a time dependence in the derivative payment structure decaying the payments for large time values. This is related to another market phenomenon – the investors prefer to get their profit earlier forsaking some eventual additional payments in the future. The discount factor has another significance in the absence of maturity. It limits the exercise moment making exercising after a large time unattractive. In that way the discount factor provides a benefit for earlier exercising. Obviously, the larger discount rate

[1] This method is a particular case of the stochastic differential games examined in [2].

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leads to a larger benefit. There exists another major importance of the discount factor. We prove a proposition which shows that a model with continuous dividend payments can be described as a discounted non-dividend model with changed parameters. This means that the lapse of a dividend payment is not important in the presence of discounting. Thus we can assume that the underlying asset has not dividend payments. Note that we do not impose positiveness of the risk-free rate.

We shall denote by $N_1(t, x)$ and $N_2(t, x)$ the buyer's and seller's payment structures at the moment t assuming that the underlying asset has value $S_t = x$. For a game call option with strike price K they are $N_1(t,x) = e^{-\lambda t}(x-K)^+$ and $N_2(t,x) = e^{-\lambda t}\left[(x-K)^+ + \eta\right]$, respectively.

Very important characteristics of the game options are the exercise regions. The set $\mathbb{R}^+ \times \mathbb{R}^+$ which consists of all possible points (t, S_t) can be divided into three parts – (A) the buyer's exercise region, in which immediate exercising is optimal for the buyer, (B) the seller's exercise region, in which immediate canceling is optimal for the seller, and (C) the continuation region which gives for both the buyer and the seller opportunities for a larger future profit (a smaller loss for the seller). The boundaries between the regions are called optimal exercise boundaries. Since the asset price is driven by a Markov process, its future behavior depends only on the current value. The form of the payment structures leads to $N_1(t+s,x) = e^{-\lambda s}N_1(t,x)$ and $N_2(t+s,x) = e^{-\lambda s}N_2(t,x)$. Also, since we have not a maturity date, both participants are not threatened by a forced exchange. All these facts mean that the optimal strategies of the buyer and the seller do not depend on the current moment, but only on the asset value. This means that the exercise boundaries are some constants and we can examine only the case $t = 0$.

The form of the exercise regions and the corresponding boundaries vary significantly for different values of the parameters. We examine in details all of the different cases. For a call option the buyer's exercise region is of the form $[B, \infty)$ for some exercise boundary B. In the undiscounted case, $\lambda = 0$, we prove a proposition which has an analogue for the American style options. It says that the buyer's exercise region is empty, i.e. $B = \infty$. We prove also that the seller's exercise region is $[K, \infty)$, whereas the continuation region is $(0, K)$. Otherwise, if $\lambda > 0$, then $B < \infty$. There are three cases for the seller's exercise region – $[K, A]$, $\{K\}$, and the empty set. Of course, they lead to a different form of the continuation region. For every case we determine the corresponding values of the parameters.

Our approach for deriving the optimal boundaries is based on a special kind of American style financial derivatives with stochastic maturities. For a fixed stopping time ζ_1 we define a derivative which obligates the seller to pay an amount $N_1\left(\zeta_1, S_{\zeta_1}\right)$ when the stopping time happens. The seller has the right to cancel the derivative paying a larger amount $N_2(t, S_t)$. We shall denote by $A(\zeta_1)$ its optimal strategy. Analogously, for a stopping time ζ_2 we can define a financial derivative, which obligates the seller to pay amount $N_2\left(\zeta_2, S_{\zeta_2}\right)$ at the stopping time. In this case the buyer has the right to exercise earlier receiving amount $N_1(t, S_t)$. We shall denote by $B(\zeta_2)$ its optimal strategy. We look for a stopping time ζ, which satisfies $A(B(\zeta)) = \zeta$. In that way the optimal strategies for the seller and the buyer will be ζ and $B(\zeta)$, respectively. We have to note that most of the derived results for the exercise regions are true when the maturity is finite too.

Though these results are true for an arbitrary Feller–Markov process, we assume that the underlying asset is a geometric Brownian motion. Once we had derived the exercise regions, we can calculate the game option prices using the first hitting properties of the Brownian motion.

The paper is organized as follows. In Section 2 we provide the base we shall use later. In Section 3 we derive the exercise regions. We evaluate the discounted game call options in Section 4 and we take a special attention to the undiscounted case in Section 5. We give some numerical examples in Section 6. Some propositions related to the first exit time of the Brownian motion are presented in Appendix A. In Appendix B we prove that the equations which determine the optimal boundary values have unique solutions.

2. Preliminaries

Let the asset price S_t be a Feller–Markov process under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$ which satisfies the usual conditions – the filtration is right continuous and complete. The measure Q is assumed to be risk neutral. Suppose also that the risk-free rate is a constant r. Let the discount factor be $e^{-\lambda t}$ for a positive constant $\lambda > 0$. We impose the condition, that the total discount rate is positive, $r + \lambda > 0$. Note that in the undiscounted case $\lambda = 0$, we have that the risk free rate is positive, $r > 0$. Let T be a maturity date and the set $\mathcal{T}_{[t,T]}$ consists of all stopping times with values between t and T. Let $\tau_1,\ \tau_2 \in \mathcal{T}_{[0,T]}$ be the optimal stopping times for the buyer and the seller, respectively. If the buyer stops the derivative, he receives amount $N_1\left(\tau_1, S_{\tau_1}\right)$, whereas if the seller cancels first, he has to pay amount $N_2\left(\tau_2, S_{\tau_2}\right)$.[2] It is natural to assume that $N_1(t, x) < N_2(t, x)$. Let $\tau = \tau_1 \wedge \tau_2$ be the end of the game option life. Hence, the payment at this moment can be written as

$$N(\tau, S_\tau) = N_1(\tau_1, S_{\tau_1}) I_{\tau_1 \le \tau_2} + N_2(\tau_2, S_{\tau_2}) I_{\tau_2 < \tau_1}. \tag{2.1}$$

Kifer [13] examines the case, in which for $\tau_1 = \tau_2$ the payment is $N_3\left(\tau_3, S_{\tau_3}\right)$ and it differs from $N_1\left(\tau_1, S_{\tau_1}\right)$ and $N_2\left(\tau_2, S_{\tau_2}\right)$. The necessary requirement is

$$N_1(\tau, S_\tau) \le N_3(\tau, S_\tau) \le N_2(\tau, S_\tau). \tag{2.2}$$

Thus Eq. (2.1) can be written as

$$N(\tau, S_\tau) = N_1(\tau_1, S_{\tau_1}) I_{\tau_1 < \tau_2} + N_2(\tau_2, S_{\tau_2}) I_{\tau_2 < \tau_1} + N_3(\tau, S_\tau) I_{\tau_1 = \tau_2}. \tag{2.3}$$

We have to impose another requirement

$$\mathrm{E}\left[\sup_{0 \le t \le T} \left(|N_1(t, S_t)| + |N_2(t, S_t)| + |N_3(t, S_t)|\right)\right] < \infty. \tag{2.4}$$

Let us define

$$\begin{aligned} \overline{Y}_t &= \operatorname*{ess\,inf}_{\tau_2 \in \mathcal{T}_{[t,T]}} \operatorname*{ess\,sup}_{\tau_1 \in \mathcal{T}_{[t,T]}} E\left[e^{-r(\tau - t)} N(\tau, S_\tau) \middle| \mathcal{F}_t\right] \\ \underline{Y}_t &= \operatorname*{ess\,sup}_{\tau_1 \in \mathcal{T}_{[t,T]}} \operatorname*{ess\,inf}_{\tau_2 \in \mathcal{T}_{[t,T]}} E\left[e^{-r(\tau - t)} N(\tau, S_\tau) \middle| \mathcal{F}_t\right]. \end{aligned} \tag{2.5}$$

In [12] and [13] is proved the following theorem.

Theorem 2.1. *If conditions (2.2) and (2.4) are satisfied, then*

$$Y_t = \overline{Y}_t \equiv \underline{Y}_t. \tag{2.6}$$

Moreover, the optimal stopping times are given by

$$\begin{aligned} \tau_1 &= \inf\{s : t \le s,\ Y_s \le N_1(s, S_s)\} \\ \tau_2 &= \inf\{s : t \le s,\ Y_s \ge N_2(s, S_s)\}. \end{aligned} \tag{2.7}$$

From now on we shall assume for simplicity that $N_1\left(\tau_1, S_{\tau_1}\right) \equiv N_3\left(\tau_1, S_{\tau_1}\right)$. Let the penalty be the constant η. The payments for the discounted game call options are

$$\begin{aligned} N_1(t,x) &= e^{-\lambda t}(x-K)^+ \\ N_2(t,x) &= e^{-\lambda t}\left((x-K)^+ + \eta\right). \end{aligned} \tag{2.8}$$

If the discount factor is zero, $\lambda = 0$, we reach to the ordinary game options.

[2] Sawaki et al. [23] introduce an additional term, which can be treated as a coupon payment.

We shall use several times the following natural proposition. It presents the fact that if the asset starts from a higher value, then the expected future value has to be higher too.

Proposition 2.1. *Let ζ be a bounded stopping time. If $0<y<x$, then $E^{t,y}\left[e^{-r\zeta}S_\zeta\right] < E^{t,x}\left[e^{-r\zeta}S_\zeta\right]$.*

Proof. The proof is easy and can be found in [26], Proposition 2.1. □

Now we shall prove a proposition which explains how the option prices depend on the time.

Proposition 2.2. *Let the function $Y(t, x)$ defines the price of a live perpetual game option in moment $t>0$ assuming that the current asset price is $S_t = x$. Then $Y(t,x) = e^{-\lambda t}Y(0,x)$.*

Proof. Since the asset price is driven by a Markov process and the stopping times are related to first hitting to some levels, we have for the price of a live option at time $t>0$

$$\begin{aligned} Y(t,x) &= E^{t,x}\left[e^{-r(\tau-t)}(N_1(\tau_1,S_{\tau_1})I_{\tau_1<\tau_2} + N_2(\tau_2,S_{\tau_2})I_{\tau_2<\tau_1})\right] \\ &= e^{-\lambda t}E^{t,x}\left[e^{-r(\tau-t)}(N_1(\tau_1-t,S_{\tau_1})I_{\tau_1<\tau_2} + N_2(\tau_2-t,S_{\tau_2})I_{\tau_2<\tau_1})\right] \\ &= e^{-\lambda t}Y(0,x). \end{aligned} \tag{2.9}$$

□

From now on we shall assume that $t=0$. Finally, we shall prove that the existence of a dividend payment is not essential in the model.

Proposition 2.3. *The model with a continuous dividend payment can be written in the terms of a non-dividend model.*

Proof. Suppose that the dividend rate is δ. We know that $e^{-(r-\delta)t}S_t$ is a Q-martingale. Let the derivative payment at the stochastic maturity date τ be of the form $e^{-\lambda\tau}C(\tau,S_\tau)$. Thus the derivative price should be

$$Y(r,\lambda,\delta) = E\left[e^{-r\tau}\left(e^{-\lambda\tau}C(\tau,S_\tau)\right)\right]. \tag{2.10}$$

Let us examine a model in which the risk free and discount rates are $\overline{r} = r-\delta$ and $\overline{\lambda} = \lambda+\delta$. Since $e^{-(r-\delta)t}S_t$ is a Q-martingale, the dividend rate has to be zero w.r.t. this model. Therefore

$$\begin{aligned} Y\left(\overline{r},\overline{\lambda},0\right) &= E\left[e^{-\overline{r}\tau}\left(e^{-\overline{\lambda}\tau}C(\tau,S_\tau)\right)\right] \\ &= E\left[e^{-r\tau}\left(e^{-\lambda\tau}C(\tau,S_\tau)\right)\right] \equiv Y(r,\lambda,\delta). \end{aligned} \tag{2.11}$$

This shows that a model with a continuous dividend can be presented as a model without dividend, but with changed values of the risk free rate and the discount factor. □

3. Exercise regions

The exercise region consists of two parts – one for the seller, Υ^s, and another for the buyer, Υ^b. We also have to define the following optimal stopping times.

Definition 3.1.

1. For every stopping time $\zeta_1 \in \mathcal{T}_{[t,T]}$ we define $A(\zeta_1;\, x)$ as the stopping time which minimizes

$$E^{t,x}\left[\begin{array}{l} e^{-r(\zeta_1-t)}N_1\left(\zeta_1,S_{\zeta_1}\right)I_{\zeta_1\le A(\zeta_1;\cdot)} \\ +e^{-r(A(\zeta_1;\cdot)-t)}N_2\left(A(\zeta_1;\cdot),S_{A(\zeta_1;\cdot)}\right)I_{A(\zeta_1;\cdot)<\zeta_1} \end{array}\right]. \tag{3.1}$$

$A(\zeta_1;\, x)$ can be viewed as the seller's optimal strategy if the buyer has a strategy ζ_1.

2. Analogously, we define the buyer's optimal strategy, $B(\zeta_2;\, x)$, as the stopping time which maximizes

$$E^{t,x}\left[\begin{array}{l} e^{-r(B(\zeta_2;\cdot)-t)}N_1\left(B(\zeta_2;\cdot),S_{B(\zeta_2;\cdot)}\right)I_{B(\zeta_2;\cdot)\le\zeta_2} \\ +e^{-r(\zeta_2-t)}N_2\left(\zeta_2,S_{\zeta_2}\right)I_{\zeta_2<B(\zeta_2;\cdot)} \end{array}\right]. \tag{3.2}$$

We are ready now to define the exercise regions.

Definition 3.2.

1. The point $(t,\, x)\in\Upsilon^b$ if for every stopping time $\zeta_1 \in \mathcal{T}_{[t,T]}$,

$$N_1(t,x) \ge E^{t,x}\left[\begin{array}{l} e^{-r(\zeta_1-t)}N_1\left(\zeta_1,S_{\zeta_1}\right)I_{\zeta_1\le A(\zeta_1;\cdot)} \\ +e^{-r(A(\zeta_1;\cdot)-t)}N_2\left(A(\zeta_1;\cdot),S_{A(\zeta_1;\cdot)}\right)I_{A(\zeta_1;\cdot)<\zeta_1} \end{array}\right]. \tag{3.3}$$

2. The point $(t,\, x)\in\Upsilon^s$ if for every stopping time $\zeta_2 \in \mathcal{T}_{[t,T]}$,

$$N_2(t,x) \le E^{t,x}\left[\begin{array}{l} e^{-r(B(\zeta_2;\cdot)-t)}N_1\left(B(\zeta_2;\cdot),S_{B(\zeta_2;\cdot)}\right)I_{B(\zeta_2;\cdot)\le\zeta_2} \\ +e^{-r(\zeta_2-t)}N_2\left(\zeta_2,S_{\zeta_2}\right)I_{\zeta_2<B(\zeta_2;\cdot)} \end{array}\right]. \tag{3.4}$$

3. The continuation region is $\overline{\Upsilon} = \{[0,T]\times\mathbb{R}^+\}/\left\{\Upsilon^b\cup\Upsilon^s\right\}$, i.e. these points which give to the buyer and the seller opportunities for a future larger profit (a smaller loss for the seller).

Now we shall prove a series of propositions which give the form of the early exercise regions.

3.1. Undiscounted game call options

Note that since $r+\lambda>0$, we have that $r>0$. The following proposition has an analogue for American call options.

Proposition 3.1. *If the discount factor is zero, $\lambda=0$, and the buyer's payment is $N_1(t,x) = (x-K)^+$, then its exercise region is empty – $\Upsilon^b = \emptyset$.*

Proof. Suppose that $(t,\, x)\in\Upsilon^b$ and the stopping time ζ is finite. Obviously $x=S_t>K$ and therefore, using the martingality of $e^{-rt}S_t$, we see

$$\begin{aligned} & E^{t,x}\left[e^{-r\zeta}N_1\left(\zeta,S_\zeta\right)I_{\zeta\le A(\zeta;\cdot)} + e^{-rA(\zeta;\cdot)}N_2\left(A(\zeta;\cdot),S_{A(\zeta;\cdot)}\right)I_{A(\zeta;\cdot)<\zeta}\right] \\ &\le e^{-rt}N_1(t,x) = e^{-rt}(x-K) \\ &= E^{t,x}\left[e^{-r(\zeta\wedge A(\zeta;\cdot))}S_{\zeta\wedge A(\zeta;\cdot)}\right] - Ke^{-rt} \\ &= E^{t,x}\left[e^{-r\zeta}S_\zeta I_{\zeta\le A(\zeta;\cdot)}\right] + E^{t,x}\left[e^{-rA(\zeta;\cdot)}S_{A(\zeta;\cdot)}I_{A(\zeta;\cdot)<\zeta}\right] - Ke^{-rt} \\ &< E^{t,x}\left[e^{-r\zeta}\left(S_\zeta-K\right)I_{\zeta\le A(\zeta;\cdot)} + e^{-rA(\zeta;\cdot)}\left(S_{A(\zeta;\cdot)}-K\right)I_{A(\zeta;\cdot)<\zeta}\right] \\ &\le E^{t,x}\left[e^{-r\zeta}N_1\left(\zeta,S_\zeta\right)I_{\zeta\le A(\zeta;\cdot)} + e^{-rA(\zeta;\cdot)}N_1\left(A(\zeta;\cdot),S_{A(\zeta;\cdot)}\right)I_{A(\zeta;\cdot)<\zeta}\right]. \end{aligned}$$

Since $N_1(t,\, x)\le N_2(t,\, x)$, we reach to a contradiction. □

We shall prove later that the seller's optimal boundary is infinity too.

3.2. Discounted perpetual call options

First, we shall show that for $\eta\ge K$ a perpetual game call option turns to an ordinary perpetual American call.

Proposition 3.2. *If $\eta\ge K$, then $\Upsilon^s = \emptyset$.*

Proof. Suppose that $x\in\Upsilon^s$ and therefore inequality (3.4) is satisfied for every stopping time ζ_2. Let $\zeta_2 = u>0$ be deterministic. Note that when u tends to infinity, $B(\zeta_2;\, x)$ is finite and therefore (3.4) leads to

$$\begin{aligned} 0 &\ge \lim_{u\to\infty}\left(\begin{array}{l} (x-K)^+ + \eta\left(1-e^{-(r+\lambda)u}P(u<B(u;\cdot))\right) \\ -E^x\left[^{-(r+\lambda)(u\wedge B(u;\cdot))}\left(S_{u\wedge B(u;\cdot)}-K\right)^+\right] \end{array}\right) \\ &= (x-K)^+ + \eta - E^x\left[e^{-(r+\lambda)B(u;\cdot)}\left(S_{B(u;\cdot)}-K\right)^+\right] \\ &= (x-K)^+ + \eta \\ &\quad -E^x\left[e^{-(r+\lambda)B(u;\cdot)}S_{B(u;\cdot)} - e^{-(r+\lambda)B(u;\cdot)}K + e^{-(r+\lambda)B(u;\cdot)}\left(K-S_{B(u;\cdot)}\right)^+\right] \\ &\ge \max(\eta-K,\eta-x) + E^x\left[e^{-(r+\lambda)B(u;\cdot)}\min\left(S_{B(u;\cdot)},K\right)\right]. \end{aligned}$$

Since $\eta \geq K$, the first term in the last equation is non-negative which leads to a contradiction. □

It is clear that if $x \leq K$ then $x \notin \Upsilon^b$. We shall prove the analogous proposition for the seller's region.

Proposition 3.3. *If $x < K$ then $x \notin \Upsilon^s$.*

Proof. Obviously, if the starting point x is bellow the strike K, then the strategy of the first hitting time to K leads to a better result for the seller – he will pay amount $e^{-\lambda\tau}\eta$ in a future time τ. Its present value is $E\left[e^{-r\tau}e^{-\lambda\tau}\eta\right] < \eta$, since $r + \lambda > 0$. □

The following two propositions give us other important features of the exercise regions.

Proposition 3.4. *Suppose that $x \in \Upsilon^b$ and $y > x$. Then $y \in \Upsilon^b$ too.*

Proof. Let T be a fixed finite moment. Let $\zeta \in \mathcal{T}[0,T]$ be an arbitrary stopping time and $A(\zeta; x)$ be the corresponding ζ-seller's optimal strategy. Then condition (3.3) leads to

$$E^x\left[\begin{array}{l} e^{-(r+\lambda)\zeta}\left(S_\zeta - K\right)^+ I_{\zeta \leq A(\zeta;x)} \\ + e^{-(r+\lambda)A(\zeta;x)}\left(\left(S_{A(\zeta;x)} - K\right)^+ + \eta\right) I_{A(\zeta;x)<\zeta} \end{array}\right] \leq (x - K).$$

Using Proposition 2.1 and the fact that $A(\zeta;\ y)$ minimizes Eq. (3.1) we derive

$$\begin{aligned}
& E^y\left[\begin{array}{l} e^{-(r+\lambda)\zeta}\left(S_\zeta - K\right)^+ I_{\zeta \leq A(\zeta;y)} \\ + e^{-(r+\lambda)A(\zeta;y)}\left(\left(S_{A(\zeta;y)} - K\right)^+ + \eta\right) I_{A(\zeta;y)<\zeta} \end{array}\right] - (y - K) \\
&= E^y\left[\begin{array}{l} e^{-(r+\lambda)\zeta}\left(S_\zeta - K\right)^+ I_{\zeta \leq A(\zeta;y)} \\ + e^{-(r+\lambda)A(\zeta;y)}\left(\left(S_{A(\zeta;y)} - K\right)^+ + \eta\right) I_{A(\zeta;y)<\zeta} \end{array}\right] \\
&- E^y\left[e^{-(r+\lambda)\zeta}e^{\lambda\zeta}S_\zeta I_{\zeta \leq A(\zeta;y)} + e^{-(r+\lambda)A(\zeta;y)}e^{\lambda A(\zeta;y)}S_{A(\zeta;y)} I_{A(\zeta;y)<\zeta}\right] + K \\
&= E^y\left[\begin{array}{l} e^{-(r+\lambda)\zeta} \max\left(-\left(e^{\lambda\zeta} - 1\right)S_\zeta - K, -e^{\lambda\zeta}S_\zeta\right) I_{\zeta \leq A(\zeta;y)} \\ + e^{-(r+\lambda)A(\zeta;y)} \max\left(\begin{array}{c} -\left(e^{\lambda A(\zeta;y)} - 1\right)S_{A(\zeta;y)} - K + \eta, \\ -e^{\lambda A(\zeta;y)}S_{A(\zeta;y)} + \eta \end{array}\right) I_{A(\zeta;y)<\zeta} \end{array}\right] + K \\
&\leq E^y\left[\begin{array}{l} e^{-(r+\lambda)\zeta} \max\left(-\left(e^{\lambda\zeta} - 1\right)S_\zeta - K, -e^{\lambda\zeta}S_\zeta\right) I_{\zeta \leq A(\zeta;x)} \\ + e^{-(r+\lambda)A(\zeta;x)} \max\left(\begin{array}{c} -\left(e^{\lambda A(\zeta;x)} - 1\right)S_{A(\zeta;x)} - K + \eta, \\ -e^{\lambda A(\zeta;x)}S_{A(\zeta;x)} + \eta \end{array}\right) I_{A(\zeta;x)<\zeta} \end{array}\right] + K \\
&< E^x\left[\begin{array}{l} e^{-(r+\lambda)\zeta} \max\left(-\left(e^{\lambda\zeta} - 1\right)S_\zeta - K, -e^{\lambda\zeta}S_\zeta\right) I_{\zeta \leq A(\zeta;x)} \\ + e^{-(r+\lambda)A(\zeta;x)} \max\left(\begin{array}{c} -\left(e^{\lambda A(\zeta;x)} - 1\right)S_{A(\zeta;x)} - K + \eta, \\ -e^{\lambda A(\zeta;x)}S_{A(\zeta;x)} + \eta \end{array}\right) I_{A(\zeta;x)<\zeta} \end{array}\right] + K \\
&= E^x\left[\begin{array}{l} e^{-(r+\lambda)\zeta}\left(S_\zeta - K\right)^+ I_{\zeta \leq A(\zeta;x)} \\ + e^{-(r+\lambda)A(\zeta;x)}\left(\left(S_{A(\zeta;x)} - K\right)^+ + \eta\right) I_{A(\zeta;x)<\zeta} \end{array}\right] - (x - K) \leq 0.
\end{aligned}$$

After taking $T \to \infty$ we finish the proof. □

Proposition 3.5. *Suppose that $x \in \Upsilon^s$ and $K < y < x$. Then $y \in \Upsilon^s$ too.*

Proof. Let again T be a fixed finite moment, $\zeta \in \mathcal{T}[0,T]$ be arbitrary, and $B(\zeta; x)$ be the ζ-buyer's optimal strategy. Thus condition (3.4) leads to

$$E^x\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\left(S_{B(\zeta;x)} - K\right)^+ I_{B(\zeta;x)\leq\zeta} \\ + e^{-(r+\lambda)\zeta}\left(\left(S_\zeta - K\right)^+ + \eta\right) I_{\zeta<B(\zeta;x)} \end{array}\right] \geq (x - K + \eta).$$

Analogously to the proof of Proposition 3.4, we derive

$$\begin{aligned}
& E^y\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)}\left(S_{B(\zeta;y)} - K\right)^+ I_{B(\zeta;y)\leq\zeta} \\ + e^{-(r+\lambda)\zeta}\left(\left(S_\zeta - K\right)^+ + \eta\right) I_{\zeta<B(\zeta;y)} \end{array}\right] - (y - K + \eta) \\
&= E^y\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)}\left(S_{B(\zeta;y)} - K\right)^+ I_{B(\zeta;y)\leq\zeta} \\ + e^{-(r+\lambda)\zeta}\left(\left(S_\zeta - K\right)^+ + \eta\right) I_{\zeta<B(\zeta;y)} \end{array}\right] \\
&- E^y\left[e^{-(r+\lambda)B(\zeta;y)}e^{\lambda B(\zeta;y)}S_{B(\zeta;y)} I_{B(\zeta;y)\leq\zeta} + e^{-(r+\lambda)\zeta}e^{\lambda\zeta}S_\zeta I_{\zeta<B(\zeta;y)}\right] \\
&+ K - \eta \\
&= E^y\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)} \max\left(\begin{array}{c} -\left(e^{\lambda B(\zeta;y)} - 1\right)S_{B(\zeta;y)} - K, \\ -e^{\lambda B(\zeta;y)}S_{B(\zeta;y)} \end{array}\right) I_{B(\zeta;y)\leq\zeta} \\ + e^{-(r+\lambda)\zeta} \max\left(\begin{array}{c} -\left(e^{\lambda\zeta} - 1\right)S_\zeta - K + \eta, \\ -e^{\lambda\zeta}S_\zeta + \eta \end{array}\right) I_{\zeta<B(\zeta;x)} \end{array}\right] \\
&+ K - \eta \\
&\geq E^y\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \max\left(\begin{array}{c} -\left(e^{\lambda B(\zeta;x)} - 1\right)S_{B(\zeta;x)} - K, \\ -e^{\lambda B(\zeta;x)}S_{B(\zeta;x)} \end{array}\right) I_{B(\zeta;x)\leq\zeta} \\ + e^{-(r+\lambda)\zeta} \max\left(\begin{array}{c} -\left(e^{\lambda\zeta} - 1\right)S_\zeta - K + \eta, \\ -e^{\lambda\zeta}S_\zeta + \eta \end{array}\right) I_{\zeta<B(\zeta;x)} \end{array}\right] \\
&+ K - \eta \\
&> E^x\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \max\left(\begin{array}{c} -\left(e^{\lambda B(\zeta;x)} - 1\right)S_{B(\zeta;x)} - K, \\ -e^{\lambda B(\zeta;x)}S_{B(\zeta;x)} \end{array}\right) I_{B(\zeta;x)\leq\zeta} \\ + e^{-(r+\lambda)\zeta} \max\left(\begin{array}{c} -\left(e^{\lambda\zeta} - 1\right)S_\zeta - K + \eta, \\ -e^{\lambda\zeta}S_\zeta + \eta \end{array}\right) I_{\zeta<B(\zeta;x)} \end{array}\right] \\
&+ K - \eta \\
&= E^x\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\left(S_{B(\zeta;x)} - K\right)^+ I_{B(\zeta;x)\leq\zeta} \\ + e^{-(r+\lambda)\zeta}\left(\left(S_\zeta - K\right)^+ + \eta\right) I_{\zeta<B(\zeta;x)} \end{array}\right] - (x - K + \eta) \geq 0.
\end{aligned}$$

We finish the proof by taking $T \to \infty$. □

Proposition 3.4 prompts us to expect that the buyer's exercise region is of the form $\Upsilon^b = [B, \infty)$. If the discount factor is zero, Proposition 3.1 says that $B = \infty$. Proposition 3.5 shows that the seller's exercise region has one of the three forms $\Upsilon^s = [K, A]$, $\Upsilon^s = \{K\}$, or $\Upsilon^s = \emptyset$.

We shall prove the following proposition for the seller's optimal region.

Proposition 3.6. *If the risk free rate is negative, $r < 0$, then the seller's optimal region is either empty, $\Upsilon^s = \emptyset$, or consists only of the point $\{K\}$, $\Upsilon^s = \{K\}$.*

Proof.

1. If the initial asset price x is less than the strike, $x < K$, then it is better for the seller to pay amount $e^{-\lambda\zeta}\eta$ in a future moment instead to pay now.[3] This is true because its present value is $E\left[e^{-r\zeta}e^{-\lambda\zeta}\eta\right] < \eta$, because $r + \lambda > 0$. Therefore $(0, K) \cap \Upsilon^s = \emptyset$.
2. Let $x > K$ and suppose that $x \in \Upsilon^s$. Therefore $x \notin \Upsilon^b$. Proposition 3.4 gives us that there exists a constant $K_1 > x$, such that $K_1 \notin \Upsilon^b$ too. Let ζ be the first exit time from the strip $(K_1,$

[3] ζ is the first hitting time to the strike.

K). Proposition 3.4 gives us also that $B(\zeta; x) > \zeta$. Thus we have

$$E^x\Big[e^{-(r+\lambda)\zeta}\big(S_\zeta - K + \eta\big)I_{\zeta \le B(\zeta;x)} + e^{-(r+\lambda)B(\zeta;x)}\big(S_\zeta - K\big)I_{B(\zeta;x)<\zeta}\Big]$$
$$\le E^x\Big[e^{-r\zeta}\big(S_\zeta - K + \eta\big)\Big] < x - K + \eta,$$

which shows that $x \notin \Upsilon^s$.[4]

□

Now we shall examine the limiting case $\eta = 0$. Note that Propositions 3.4 and 3.5 still hold. We shall prove the following proposition.

Proposition 3.7. *Let $\eta = 0$ and $x > K$. Then it is optimal for one of the buyer or the seller to exercise the option immediately.*

Proof. Let x be the initial value of the underlying asset and A and B be such that $B > x > A \ge K$. Let us examine a financial derivative related to a first exit time from the strip (A, B). Let it pays $B - K$ if the exit happens from the upper boundary and $A - K$, otherwise. Let the function $f(A, B, x)$ defines its price. Let for a fixed A, the boundary $B(A, x)$ maximizes the derivative value in the interval $[x, \infty)$. Similarly, for a fixed B we define $A(B, x)$ as the value which minimizes the derivative price on the interval $[K, x]$.

Suppose that $x \notin \Upsilon^b$. Using Proposition 3.4, we conclude that there exists $\overline{B} > x$ such that

$$f\big(A(\overline{B}, x), \overline{B}, x\big) > x - K. \tag{3.5}$$

The definition of the function $A(B, x)$ shows that

$$f\big(A, \overline{B}, x\big) \ge f\big(A(\overline{B}, x), \overline{B}, x\big) > x - K \tag{3.6}$$

for every $A \in [K, x)$. The definition of the function $B(A, x)$ leads to

$$f(A, B(A, x), x) \ge f\big(A, \overline{B}, x\big) > x - K, \tag{3.7}$$

which shows that $x \in \Upsilon^s$. □

4. Pricing

From now on we shall assume that the Feller–Markov process is a Brownian motion and the asset price is the log-normal process

$$dS_t = rS_t + \sigma S_t dB_t. \tag{4.1}$$

In Section 3.2 we had proved that the buyer's exercise region is of the form $\Upsilon^b = [B, \infty)$. We have three cases for the seller's exercise region – $\Upsilon^s = [K, A)$, $\Upsilon^s = \{K\}$, or $\Upsilon^s = \emptyset$. Thus we have to recognize which of these cases holds and after that we have to find the values of A and B.

Suppose that we are in the first case – the seller's boundary is larger than the strike, $A > K$. If the initial asset price is between the barriers, $x \in [A, B]$, then the game option pricing is related to the first exit time of the Brownian motion from a strip. Let τ_A and τ_B be the first hitting times of the asset to the values A and B, respectively. Therefore the exit moment from the continuation region is $\tau = \tau^A \wedge \tau^B$. In the terms of the Brownian motion, τ^A and τ^B are the first hitting times of a Brownian motion with drift

$$\psi = \frac{r}{\sigma} - \frac{\sigma}{2} \tag{4.2}$$

to the values

$$\begin{aligned}\overline{A} &= \tfrac{\ln A - \ln x}{\sigma} < 0\\ \overline{B} &= \tfrac{\ln B - \ln x}{\sigma} > 0.\end{aligned} \tag{4.3}$$

Hence, the price of the game call option has to be

$$\begin{aligned}f(A, B, x) &= E\begin{bmatrix} e^{-(r+\lambda)\tau^B}(S_{\tau^B} - K)I_{\tau^B \le \tau^A} \\ +e^{-(r+\lambda)\tau^A}(S_{\tau^A} - K + \eta)I_{\tau^A < \tau^B}\end{bmatrix}\\ &= (B - K)E^x\Big[e^{-(r+\lambda)\tau^B}I_{\tau^B \le \tau^A}\Big]\\ &+(A - K + \eta)E^x\Big[e^{-(r+\lambda)\tau^A}I_{\tau^A < \tau^B}\Big].\end{aligned} \tag{4.4}$$

Let us define

$$\begin{aligned}\mu &:= \tfrac{\psi}{\sigma}\\ c &:= \sqrt{\mu^2 + 2\tfrac{r+\lambda}{\sigma^2}}\\ &= \sqrt{\left(\tfrac{r}{\sigma^2} - \tfrac{1}{2}\right)^2 + 2\tfrac{r+\lambda}{\sigma^2}}.\end{aligned} \tag{4.5}$$

We can see immediately that

$$c \ge \mu + 1 \tag{4.6}$$

and we have equality when $\lambda = 0$ and $r > 0$. Using formulas (A.3) and (A.4) for the Laplace transforms of the first exit time of the Brownian motion from a strip[5] we see that Eq. (4.4) turns to

$$\begin{aligned}f(A, B, x) &= (A - K + \eta)e^{\psi\overline{A}}\tfrac{\sinh(\sigma c\overline{B})}{\sinh(\sigma c(\overline{B} - \overline{A}))}\\ &+(B - K)e^{\psi\overline{B}}\tfrac{\sinh(-\sigma c\overline{A})}{\sinh(\sigma c(\overline{B} - \overline{A}))}\\ &= (A - K + \eta)e^{(c-\mu)(\ln x - \ln A)}\tfrac{e^{2c(\ln B - \ln x)} - 1}{e^{2c(\ln B - \ln A)} - 1}\\ &+(B - K)e^{(c+\mu)(\ln B - \ln x)}\tfrac{e^{2c(\ln x - \ln A)} - 1}{e^{2c(\ln B - \ln A)} - 1}\\ &= (A - K + \eta)\left(\tfrac{x}{A}\right)^{c-\mu}\tfrac{\left(\frac{B}{x}\right)^{2c} - 1}{\left(\frac{B}{A}\right)^{2c} - 1} + (B - K)\left(\tfrac{B}{x}\right)^{c+\mu}\tfrac{\left(\frac{x}{A}\right)^{2c} - 1}{\left(\frac{B}{A}\right)^{2c} - 1}\\ &= (A - K + \eta)\left(\tfrac{A}{x}\right)^{c+\mu}\tfrac{B^{2c} - x^{2c}}{B^{2c} - A^{2c}} + (B - K)\left(\tfrac{B}{x}\right)^{c+\mu}\tfrac{x^{2c} - A^{2c}}{B^{2c} - A^{2c}}.\end{aligned} \tag{4.7}$$

Suppose that B is fixed and let us change variables as $a = \frac{A}{B}$, $k = \frac{K}{B}$, $y = \frac{x}{B}$, and $\xi = \frac{\eta}{B}$. Thus we have $0 < \xi < k < a < 1$. Therefore Eq. (4.7) turns to

$$f(A, B, x) = \frac{B}{y^{c+\mu}}\frac{(a - k + \xi)a^{c+\mu}\left(1 - y^{2c}\right) + (1 - k)\left(y^{2c} - a^{2c}\right)}{1 - a^{2c}}. \tag{4.8}$$

Let $q = c + \mu$ and $p = 2c$. Inequality (4.6) gives us that $p \ge q + 1$ as equality is reached in the undiscounted case, $\lambda = 0$. We shall find which value of the variable a minimizes the function

$$\begin{aligned}g(a; y) &= \frac{(a - k + \xi)a^q(1 - y^p) + (1 - k)(y^p - a^p)}{1 - a^p}\\ &= \frac{-a^p(1-k) + a^{q+1}(1 - y^p) - a^q(k - \xi)(1 - y^p) + y^p(1 - k)}{1 - a^p}.\end{aligned} \tag{4.9}$$

Its a-derivative is equal to

$$g_a(a; y) = \frac{1 - y^p}{(1 - a^p)^2}a^{q-1}\begin{bmatrix} a^{p+1}(p - q - 1) - a^p(k - \xi)(p - q)\\ -a^{p-q}p(1 - k) + a(q + 1) - q(k - \xi)\end{bmatrix}. \tag{4.10}$$

In Appendix B.1 is proven that the equation

$$\begin{aligned}&a^{p+1}(p - q - 1) - a^p(k - \xi)(p - q) - a^{p-q}p(1 - k) + a(q + 1)\\ &-q(k - \xi) = 0\end{aligned} \tag{4.11}$$

has just one solution in the interval (0, 1). Note that its value is independent of the variable y. Since the function $g_a(a; y)$ is negative before this value and positive after that, the function $g(a; y)$ has a minimum. Let us denote by $a(B)$ just this value. Note that it depends on B by the variables k and ξ. Thus we can conclude that if the buyer has a strategy to exercise when the asset hits the value B, then the seller's optimal stopping time is the first hitting time to the value $A = a(B)B$.

Now we turn to the buyer's exercise boundary. Suppose that A is fixed and let us change variables as $b = \frac{B}{A}$, $k = \frac{K}{A}$, $y = \frac{x}{A}$, and $\xi = \frac{\eta}{A}$. Therefore $0 < \xi < k < 1 < b$. Thus Eq. (4.7) turns to

$$f(A, B, x) = \frac{A}{y^q}\frac{(1 - k + \xi)(b^p - y^p) + (b - k)b^q(y^p - 1)}{b^p - 1}. \tag{4.12}$$

[4] We used that the discounted asset price is a martingale.

[5] (3.0.5 a & b) from Borodin and Salminen [3]

Let us define the function $g(\cdot;\ \cdot)$ as

$$g(b;y)=\frac{(1-k+\xi)(b^p-y^p)+(b-k)b^q(y^p-1)}{b^p-1}. \tag{4.13}$$

Its derivative is

$$g_b(b;y)=\frac{y^p-1}{(b^p-1)^2}b^{q-1}\begin{bmatrix}-b^{p+1}(p-q-1)+b^pk(p-q)\\+b^{p-q}p(1-k+\xi)-b(q+1)+qk\end{bmatrix}. \tag{4.14}$$

In Appendix B.2 is proven that the equation

$$-b^{p+1}(p-q-1)+b^pk(p-q)+b^{p-q}p(1-k+\xi)-b(q+1)+qk=0 \tag{4.15}$$

has only one solution in the interval $(1, \infty)$. For a fixed value A let us denote it by $b(A)$. Analogously as above, we can conclude that the function $g(b;\ y)$ has a maximum. Therefore if the seller's strategy is the first hitting time to the value A, then the buyer's optimal strategy is the first hitting time to the value $B=b(A)A$. Thus we can find the seller's boundary as the solution of the equation

$$b(y)a(yb(y))=1. \tag{4.16}$$

The buyer's boundary is derived as $B=b(A)A$, where A is the solution of Eq. (4.16). Suppose first that $A>K$. Hence, if the initial asset value is between the boundaries, $x\in[A,\ B]$, then the option price is given by Eq. (4.7). If $x\in\Upsilon^s\equiv[K,\ A)$, the price turns to $x-K+\eta$. Otherwise, if $x\in\Upsilon^b\equiv[B,\ \infty)$, then the price is $x-K$. If $x<K$, then the stopping time is the first hitting time to the strike, which leads to the option price

$$E^x\left[e^{-(r+\lambda)\tau}\left((S_\tau-K)^++\eta\right)I_{\tau<\infty}\right]=\eta E^x\left[e^{-(r+\lambda)\tau}I_{\tau<\infty}\right]=\eta\left(\frac{x}{K}\right)^\gamma \tag{4.17}$$

for

$$\gamma=p-q\equiv c-\mu. \tag{4.18}$$

We used above Eq. (A.1) from Proposition A.1.

Note that to keep the differentiability of function (4.7) we use $A-K+\eta$ instead $(A-K)^++\eta$ for the seller's payment. This may lead to a result less than the strike, $A<K$, and this will be incorrect. Hence, we have to proceed in a different way. Proposition 3.3 shows that the points under the strike are not optimal for the seller. Thus we have to examine only the case when the seller's exercise region consists only of the strike, $\Upsilon^s=\{K\}$. In that case the seller has the alternatives (A) to cancel when the asset price hits the strike K or (B) to do nothing. In the case (B) he will wait the buyer to exercise in the optimal for him moment. Thus the game option turns to an American one. In Zaevski [26] is proven that the optimal boundary is

$$B=\frac{\gamma}{\gamma-1}K. \tag{4.19}$$

Thus if the initial asset value is $x=K$, then the option price turns to

$$\overline{\eta}=\frac{K}{\gamma}\left(\frac{\gamma-1}{\gamma}\right)^{\gamma-1}. \tag{4.20}$$

If the seller chooses the case (A) he will pay amount η. Hence, the seller's exercise region is $\Upsilon^s=\{K\}$ if $\eta<\overline{\eta}$ and $\Upsilon=\emptyset$ otherwise. If $\eta\geq\overline{\eta}$ the game option turns to an American one. Otherwise, if $\eta<\overline{\eta}$, the results are similar to these for the case $A>K$.

Finally, we summarize the obtained results for the perpetual game options in the following theorem.

Theorem 4.1. *Let $\overline{A}$ and $\overline{B}$ be the derived boundaries. Then the exercise regions and the price of the discounted perpetual game call option,Y, are*

1. *if $\overline{A}>K$, then the exercise regions are $\Upsilon^s=\left[K,\overline{A}\right]$ and $\Upsilon^b=\left[\overline{B},\infty\right)$. The game option price is*
   (a) *If $x\leq K$, then*
   $$Y=\eta\left(\frac{x}{K}\right)^\gamma. \tag{4.21}$$
   (b) *If $K<x<\overline{A}$, then*
   $$Y=x-K+\eta. \tag{4.22}$$
   (c) *If $\overline{A}\leq x\leq\overline{B}$, then*
   $$Y=\left(\overline{A}-K+\eta\right)\left(\frac{\overline{A}}{x}\right)^q\frac{\overline{B}^p-x^p}{\overline{B}^p-\overline{A}^p}+\left(\overline{B}-K\right)\left(\frac{\overline{B}}{x}\right)^q\frac{x^p-\overline{A}^p}{\overline{B}^p-\overline{A}^p}. \tag{4.23}$$
   (d) *If $\overline{B}<x$, then*
   $$Y=x-K. \tag{4.24}$$
2. *If $\overline{A}\leq K$, then*
   (a) *if $\eta\geq\overline{\eta}$, then the game option turns to an American one. The exercise regions are $\Upsilon^s=\emptyset$ and $\Upsilon^b=\left[\overline{B},\infty\right)$. The value of $\overline{B}$ is given by equation (4.19). If the initial asset value is in the buyer's exercise region, $x\in\Upsilon^b$, then the option price is given by Eq. (4.24). Otherwise, it is*
   $$Y=\left(\frac{x}{\gamma}\right)^\gamma\left(\frac{\gamma-1}{K}\right)^{\gamma-1}. \tag{4.25}$$
   (b) *If $\eta<\overline{\eta}$, then the exercise regions are $\Upsilon^s=\{K\}$ and $\Upsilon^b=\left[\overline{B},\infty\right)$. The value of $\overline{B}$ is obtained as $\overline{B}=Kb(K)$ where $b(K)$ is the zero of function (4.14).*
   If $x\leq K$, then the option price is given by formula (4.21). If $x\in\left(K,\overline{B}\right]$, then the price is given by formula (4.23) for $\overline{A}=K$. If $x>\overline{B}$, then the price is given by formula (4.24).

Remark 4.1. We have to note that Proposition 3.6 says that if the risk free rate is negative, $r<0$, then we are in the second statement of Theorem 4.1 – the seller's exercise region is either empty or consists only of the strike.

5. Perpetual game call option without discounting

We suppose now that $\lambda=0,\ r>0$ and therefore the payments are

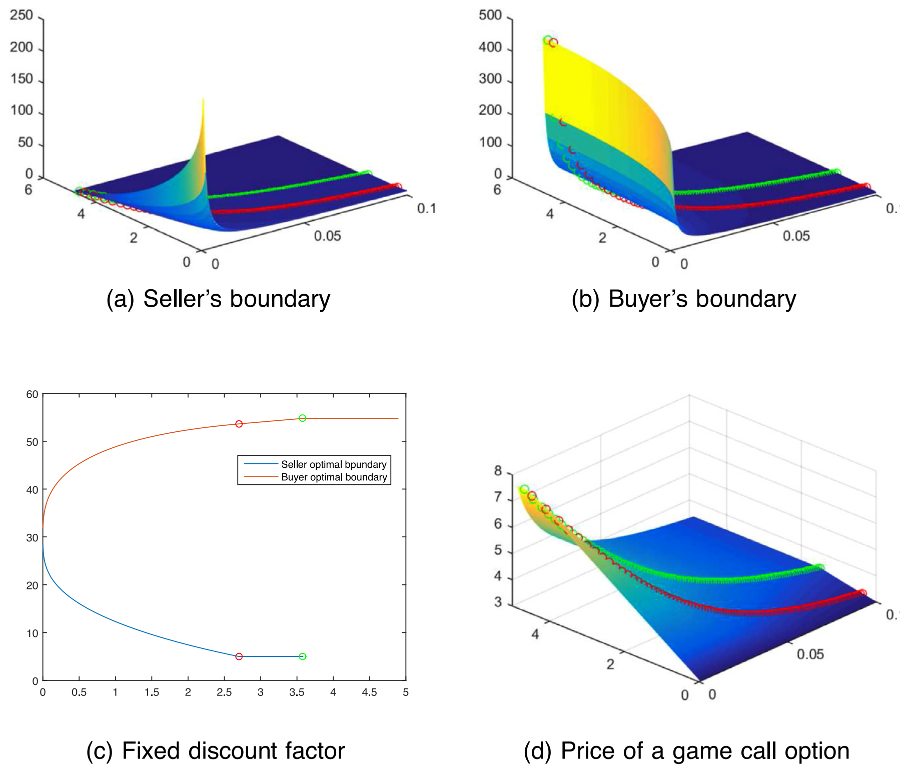
$$\begin{aligned}N_1(t,x)&=(x-K)^+\\N_2(t,x)&=(x-K)^++\eta.\end{aligned} \tag{5.1}$$

Note that Proposition 3.2 gives us again that if $\eta\geq K$ then the option turns to an ordinary American call.

Let us consider the case $\eta<K$. Proposition 3.1 gives us that the buyer's optimal region is empty and thus its boundary is infinity. Let us fix some value B and suppose that the buyer exercises when the asset hits it. We shall take the limit $B\to\infty$. We know that $p=q+1$ and the equation which determines the optimal seller's boundary turns to $h(a;\xi)=0$, where the function $h(a;\ \xi)$ is defined in (B.2). Since $\xi=\frac{\eta}{B}$ and $k=\frac{K}{B}$, its solution $\overline{a}$ is independent of B. Therefore when $B\to\infty$, the seller's boundary $A=B\overline{a}$ also tends to infinity. This means that the seller's optimal region is $\Upsilon^s\equiv(K,\ \infty)$. We have to find the value of the option in the continuation region $\{x<K\}$. Using Eq. (A.1) from Proposition A.1 for the Laplace transform of the first hitting time of a Brownian motion with drift we derive

$$Y_t=E^x\left[e^{-r\tau}\left((S_\tau-K)^++\eta\right)I_{\tau<\infty}\right]=\eta E^x\left[e^{-r\tau}I_{\tau<\infty}\right]=\frac{x\eta}{K}, \tag{5.2}$$

which corresponds to the result in [8] and Ekström and Villeneuve [9]. Of course, Eqs. (4.17) and (5.2) coincide since when $\lambda=0$, $\gamma=p-q=1$.



(a) Seller's boundary

(b) Buyer's boundary

(c) Fixed discount factor

(d) Price of a game call option

Fig. 1. Call option boundaries.

Table 1
Call option prices.

initial price $S_0 = 6$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	1.0010/1.b	2.0000/1.b	3.0000/1.b	4.0000/1.b	5.0000/1.b
$\lambda = 0.01$	1.0010/1.b	2.0000/1.b	3.0000/1.b	3.9081/2.b.2	4.3659/2.a.2
$\lambda = 0.1$	1.0010/1.b	1.7262/2.b.2	2.0536/2.a.2	2.0536/2.a.2	2.0536/2.a.2
initial price $S_0 = 7$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	2.0010/1.b	3.0000/1.b	4.0000/1.b	5.0000/1.b	6.0000/1.b
$\lambda = 0.01$	2.0010/1.b	3.0000/1.b	4.0000/1.b	4.7942/2.b.2	5.1731/2.a.2
$\lambda = 0.1$	2.0010/1.b	2.4743/2.b.2	2.6982/2.a.2	2.6982/2.a.2	2.6982/2.a.2
initial price $S_0 = 8$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	3.0010/1.b	4.0000/1.b	5.0000/1.b	6.0000/1.b	6.0000/1.b
$\lambda = 0.01$	3.0010/1.b	4.0000/1.b	4.9938/1.c	5.6707/2.b.2	5.9919/2.a.2
$\lambda = 0.1$	3.0000/1.d	3.2678/2.b.2	3.4181/2.a.2	3.4181/2.a.2	3.4181/2.a.2
initial price $S_0 = 9$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	4.0010/1.b	5.0000/1.b	6.0000/1.b	7.0000/1.b	8.0000/1.b
$\lambda = 0.01$	4.0010/1.b	5.0000/1.b	5.9591/1.c	6.5437/2.b.2	6.8212/2.a.2
$\lambda = 0.1$	4.0000/1.d	4.1172/2.b.2	4.2109/2.a.2	4.2109/2.a.2	4.2109/2.a.2
initial price $S_0 = 10$					
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	5.0010/1.b	6.0000/1.b	7.0000/1.b	8.0000/1.b	9.0000/1.b
$\lambda = 0.01$	5.0010/1.b	6.0000/1.b	6.9045/1.c	7.4166/2.b.2	7.6598/2.a.2
$\lambda = 0.1$	5.0000/1.d	5.0272/2.b.2	5.0748/2.a.2	5.0748/2.a.2	5.0748/2.a.2

6. Some numerical results

We shall examine the behavior of game call options with different parameters. We assume that the risk-free rate of return is $r = 0.05$, the volatility is $\sigma = 0.3$, the strike price is $K = 5$, and the initial asset value is $x = 8\$$. We vary the penalty η between zero and 5$ and the discount factor λ between 0.0001 and 0.1. The results are presented at Fig. 1. At sub-Fig. 1a and b we present the optimal boundaries for the seller and the buyer, respectively. We can see that when λ tends to zero, both boundaries tend to infinity.

This observation agrees with the results for the undiscounted case presented in Section 5. Note that when the penalty η is smaller, the convergence is faster and vice versa. With the red points are marked the values for the penalty η after which the boundary A falls bellow the strike K. Thus the option changes its features from the second to the first statement of Theorem 4.1. The green points are the corresponding values of $\overline{\eta}$ from Eq. (4.20). If $\eta < \overline{\eta}$, the seller's exercise region is $\Upsilon^s = \{K\}$ and the boundary is of course $A = K$. The buyer's optimal boundary is calculated as $\overline{B} = Kb(K)$ where $b(K)$ is the zero of function (4.14). Otherwise, if $\overline{\eta} \leq \eta$, the game call option turns to an American one and the seller's optimal boundary does not exist. We plotted it with a zero value.

At sub-Fig. 1c are commonly presented the seller's and buyer's optimal boundaries with a fixed discount factor $\lambda = 0.01$. The value of the penalty η for which the seller's boundary turns to the strike is marked again with a red point and it is 2.6968. The corresponding value of $\overline{\eta}$ is again marked as a green point and it is 3.5722. Proposition 3.7 has a confirmation here. When η tends to zero, the boundaries tend each to each other.

At sub-Fig. 1d is presented the behavior of the game call option prices. In Table 1 we present some particular values. We vary the discount factor as $\lambda \in \{0.001, 0.01, 0.1\}$, the penalty as $\eta \in \{0.001, 1, 2, 3, 4\}$, and the initial asset value as $x \in \{6, 7, 8, 9, 10\}$. Right to the prices we report the case of Theorem 4.1 which is actual. In the undiscounted case, examined in Section 5, the optimal boundaries are infinity. This explains why all initial values belong to the seller's exercise region when the discount factor is small, $\lambda = 0.001$. Also, for large values of the penalty η, the value of A is below the strike. For extreme large penalties the game option turns to American one.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Some propositions

Proposition A.1. *Let τ be the first hitting time of a Brownian motion with drift μ to the level a. Then the Laplace transform of its distribution is*

1. *If $a > 0$ then*

$$E\left[e^{-y\tau} I_{\tau<\infty}\right] = e^{-\left(\sqrt{\mu^2+2y}-\mu\right)a}. \tag{A.1}$$

2. *If $a < 0$ then*

$$E\left[e^{-y\tau} I_{\tau<\infty}\right] = e^{\left(\sqrt{\mu^2+2y}+\mu\right)a}. \tag{A.2}$$

Proposition A.2. *The Laplace transforms of the first exit time of a Brownian motion with drift μ from a strip (a, b) are presented by*

$$E\left[e^{-y\tau} I_{\tau=a}\right] = e^{\mu a} \frac{\sinh\left(b\sqrt{2y+\mu^2}\right)}{\sinh\left((b-a)\sqrt{2y+\mu^2}\right)} \tag{A.3}$$

$$E\left[e^{-y\tau} I_{\tau=b}\right] = e^{\mu b} \frac{\sinh\left(-a\sqrt{2y+\mu^2}\right)}{\sinh\left((b-a)\sqrt{2y+\mu^2}\right)}. \tag{A.4}$$

Appendix B. Uniqueness of the solutions

B1. Seller's boundary

Now we shall prove that derivative (4.10) has just one zero in the interval $(0, 1)$. Let us denote by $h(\cdot;\ \cdot)$ the function

$$\begin{aligned} &h(a;\xi) = a^{p+1}(p-q-1) - a^p(k-\xi)(p-q) \\ &-a^{p-q}p(1-k) + (q+1)a - q(k-\xi). \end{aligned} \tag{B.1}$$

First we shall examine the case $p = q + 1$. In fact this is the undiscounted case, $\lambda = 0$. Function (B.1) which determines the behavior of derivative (4.10) turns too

$$h(a;\xi) = -a^{q+1}(k-\xi) + a(q+1)k - q(k-\xi). \tag{B.2}$$

Its derivative

$$h_a(a;\xi) = -a^q(q+1)(k-\xi) + (q+1)k \tag{B.3}$$

is decreasing. Since $h_a(1;\xi) = (q+1)\xi > 0$, it is positive for all $a \in (0,\ 1)$. Therefore the function $h(a;\ \xi)$ is increasing. Since $h(0;\xi) = -q(k-\xi) < 0$ and $h(1;\xi) = \xi(q+1) > 0$, the solution of the equation $h(a;\xi) = 0$ is unique.

Now suppose that $p > q + 1$. The a-derivative of function (B.1) is

$$\begin{aligned} &h_a(a;\xi) = a^p(p+1)(p-q-1) - a^{p-1}p(k-\xi)(p-q) \\ &-a^{p-q-1}(p-q)p(1-k) + (q+1). \end{aligned} \tag{B.4}$$

First we shall examine in details the case $\xi = 0$ and later we shall show that the general case $\xi > 0$ is its consequence.

Suppose that $\xi = 0$. Thus derivative (B.4) turns to

$$\begin{aligned} &h_a(a;0) = a^p(p+1)(p-q-1) - a^{p-1}pk(p-q) \\ &-a^{p-q-1}(p-q)p(1-k) + (q+1). \end{aligned} \tag{B.5}$$

The corresponding second derivative is

$$\begin{aligned} h_{aa}(a;0) = {} & a^{p-1}p(p+1)(p-q-1) - a^{p-2}(p-1)pk(p-q) \\ & - a^{p-q-2}(p-q-1)(p-q)p(1-k) = a^{p-q-2} \\ & \left[\begin{array}{l} a^{q+1}p(p+1)(p-q-1) - a^q(p-1)pk(p-q) \\ -(p-q-1)(p-q)p(1-k) \end{array}\right]. \end{aligned} \tag{B.6}$$

Let the function $l(\cdot)$ be defined as

$$\begin{aligned} &l(a) = a^{q+1}p(p+1)(p-q-1) - a^q(p-1)pk(p-q) \\ &-(p-q-1)(p-q)p(1-k). \end{aligned} \tag{B.7}$$

Its a-derivative is

$$\begin{aligned} l_a(a) &= a^q(q+1)p(p+1)(p-q-1) - a^{q-1}q(p-1)pk(p-q) \\ &= a^{q-1}p[a(q+1)(p+1)(p-q-1) - q(p-1)k(p-q)]. \end{aligned} \tag{B.8}$$

Let $\overline{a}$ be the solution of the equation $l_a(a) = 0$:

$$\overline{a} = \frac{q(p-1)k(p-q)}{(q+1)(p+1)(p-q-1)} > 0. \tag{B.9}$$

We have that $l_a(a) < 0$ for $a < \overline{a}$ and $l_a(a) > 0$ for $a > \overline{a}$. We shall examine separately the cases $\overline{a} \geq 1$ and $\overline{a} < 1$.

First, suppose that $\overline{a} \geq 1$. Therefore $l_a(a) < 0$ for all $a \in (0,\ 1)$, which means that $l(a)$ is decreasing. Since $l(0) < 0$, the function $l(a)$ is negative in the whole interval $(0,\ 1)$. Of course, the same is true for the function $h_{aa}(a;\ 0)$ too. Therefore the function $h_a(a;\ 0)$ is decreasing for $a \in (0,\ 1)$. Since $h_a(1; 0) = 0$, we find that

$h_a(a;\ 0)$ is positive in the whole interval (0, 1). We have that $h_a(a;\ \xi) > h_a(a;\ 0) > 0$ too. Thus we conclude that the function $h(a;\ \xi)$ is increasing. Therefore the equation $h(a; \xi) = 0$ has just one solution, since $h(0; \xi) = -q(k-\xi) < 0$ and $h(1; \xi) = p\xi \geq 0$.

Now suppose that $\bar{a} < 1$. Therefore the function $l(a)$ has a minimum in the point $\bar{a}$. Since $l(0) < 0$, we have that $l(\bar{a}) < 0$ too. Suppose that $l(1) \leq 0$ too. Therefore the function $l(a)$ is negative for $a \in (0,\ 1)$ and we are in the previous case.

Suppose now that $l(1) > 0$. Therefore the function $l(a)$ decreases from 0 to $\bar{a}$ and then increases from $\bar{a}$ to 1. Therefore it crosses the abscissa in just one point. Also, it is negative before this point and positive after that. Hence, the function $h_a(a;\ 0)$ starts from the positive value $h_a(0; 0) = q+1$, crosses the abscissa, has a minimum and after that increases to the value $h_a(1; 0) = 0$ staying negative. Therefore the function $h(a;\ 0)$ starts from the negative value $h(0; 0) = -qk$, crosses the zero, has a maximum and then decreases to the value $h(1; 0) = 0$.

We have that $h_a(a;\ \xi) > h_a(a;\ 0)$. This means that the function $h(a;\ \xi)$ increases faster than the function $h(0;\ \xi)$ and decreases slower. Even more, it is possible $h(a;\ 0)$ to decrease, whereas $h(a;\ \xi)$ to increase. Therefore, since $h(0; 0) = -qk < -q(k-\xi) = h(0; \xi)$, we can conclude that the equation $h(a; \xi) = 0$ has only one solution, even though that the function $h(a;\ \xi)$ can be not monotone.

Remark B.1. It is possible the function $h(a;\ \xi)$ to have a local minimum near the point $a = 1$, but it has to be positive. This may happens for relatively low values of k and ξ.

B2. Buyer's boundary

We shall prove that derivative (4.14) has only one zero point in the interval $(1,\ \infty)$ except in the undiscounted case. Let us denote by $h(\cdot)$ the function

$$h(b) = -b^{p+1}(p-q-1) + b^p k(p-q) + b^{p-q} p(1-k+\xi) - b(q+1) + qk. \tag{B.10}$$

First we shall examine the undiscounted case $p = q+1$. Function (B.10) turns to

$$h(b) = b^{q+1}k - b(q+1)(k-\xi) + qk. \tag{B.11}$$

Its first derivative is

$$h_b(b) = (q+1)[b^q k - (k-\xi)] \tag{B.12}$$

and therefore it is positive for $b \in (1,\ \infty)$. Thus the function $h(b)$ is increasing. We have that $h(1) = (q+1)\xi > 0$. This means that the equation $h(b) = 0$ has no solutions which corresponds to Proposition 3.1 – the seller's exercise region is empty.

Suppose now that $p > q+1$. The derivatives of function (B.10) turn to

$$h_b(b) = -b^p(p+1)(p-q-1) + b^{p-1}pk(p-q) + b^{p-q-1}(p-q)p(1-k+\xi) - (q+1) \tag{B.13}$$

$$h_{bb}(b) = b^{p-q-2}p\begin{bmatrix} -b^{q+1}(p+1)(p-q-1) \\ +b^q(p-1)k(p-q) \\ +(p-q-1)(p-q)(1-k+\xi) \end{bmatrix}. \tag{B.14}$$

Let us denote by $l(b)$ the function

$$l(b) = -b^{q+1}(p+1)(p-q-1) + b^q(p-1)k(p-q) + (p-q-1)(p-q)(1-k+\xi). \tag{B.15}$$

Its first derivative is

$$l_b(b) = b^{q-1}[-b(q+1)(p+1)(p-q-1) + q(p-1)k(p-q)]. \tag{B.16}$$

Let us denote by $\bar{b}$ the zero of derivative (B.16)

$$\bar{b} = \frac{q(p-1)k(p-q)}{(q+1)(p+1)(p-q-1)}. \tag{B.17}$$

First, suppose that $\bar{b} \leq 1$. Then $l_b(b)$ is negative in the interval $b \in (1,\ \infty)$. Thus the function $l(b)$ is decreasing. Of course this is true for the second derivative $h_{bb}(b)$ too and thus it has no more one zero. If it has one zero, then the function $h_b(b)$ starts from the positive value $h_b(1) = p\xi(p-q) > 0$, increases to a maximum, and then decreases to minus infinity. If the second derivative $h_{bb}(b)$ has not zeros, the function $h_b(b)$ starts from the positive value $h_b(1) = p\xi(p-q) > 0$ and decreases to minus infinity. In both cases the function $h_b(b)$ is first positive, has a zero, and after that is negative. Note that $h(1) = p\xi > 0$. Using the same reasons, we conclude that the equation $h(b) = 0$ has just one solution larger than one.

Suppose now that $\bar{b} > 1$. Then the function $l(b)$ first increases in the interval $\left(1, \bar{b}\right)$ and after that decreases to minus infinity. Since the function $l(b)$ increases in the interval (0, 1) too, we have that $l(1) > l(0) = (p-q-1)(p-q)(1-k+\xi) > 0$. This means that this function is first positive, has a root, and after that is negative. The same reasons as above prove the uniqueness.

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